Queen's University

Paper

Exact Channel Simulation under Exponential Cost

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We extend results in channel simulation and exact sampling to a communication cost which is exponential in the codeword lengths.

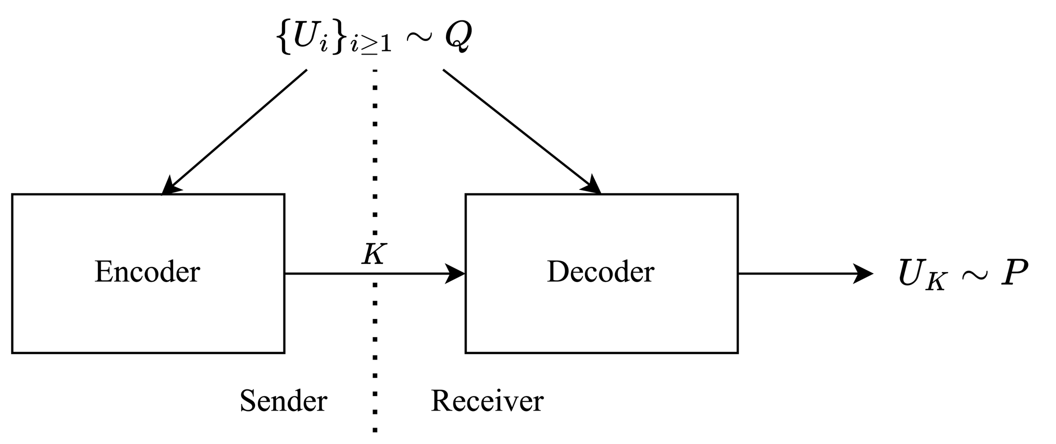
Problem Definition

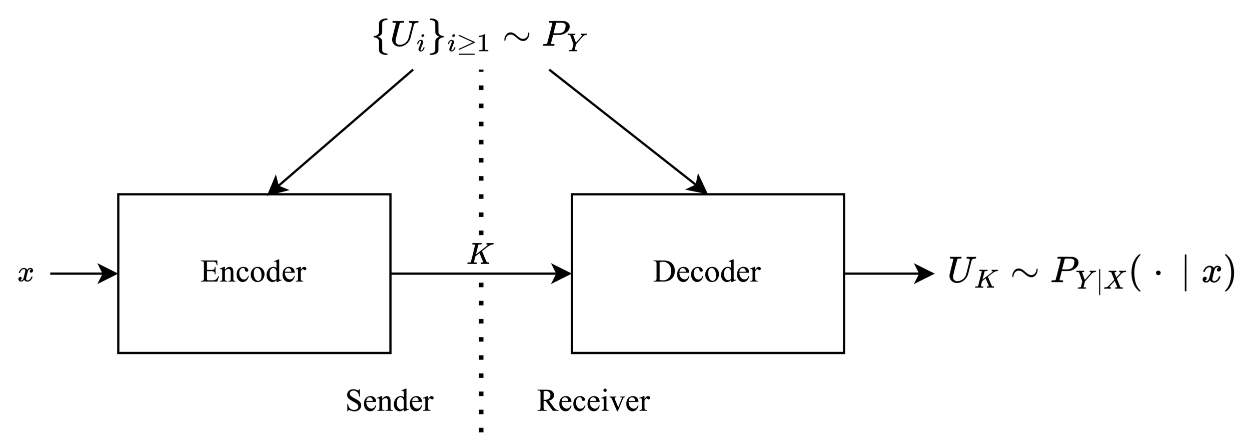
Exact Sampling

- **Input:** Distributions P, Q and shared randomness $\{U_i\}_{i\geq 1} \sim Q$
- **Output:** Index K such that $U_K \sim P$

Exact Channel Simulation

- **Input:** Joint distribution P_{XY}, shared randomness $\{U_i\}_{i\geq 1} \sim \mathsf{P}_Y$, and $x \sim \mathsf{P}_X$
- **Output:** Index K such that $U_K \sim \mathsf{P}_{Y|X}(\cdot \mid x)$





Typical goal: Minimize the expected message length $\mathbb{E}[l(\mathcal{C}(K))]$ when K is encoded by the uniquely decodable binary code $\mathcal{C}$.

Our question: What if we cared about a cost which is exponential in the message lengths? What is a lower bound on the one-shot communication cost, and can it be (almost) achieved?

Results and Bounds

We consider the Campbell cost [1] of order t:

$$L(t) = \frac{1}{t}\log(\mathbb{E}[2^{tl(\mathcal{C}(K))}]). \tag{1}$$

As $t \to 0$ in (1) we recover $\mathbb{E}[l(\mathcal{C}(K))]$. Akin to Shannon's noiseless coding theorem, Campbell [1] connected $L(t)$ with the Rényi entropy of order $\alpha = \frac{1}{1+t}$ by showing that for a discrete source X,

$$H_\alpha(X) \leq L(t) < H_\alpha(X) + 1. \tag{2}$$

Result 1: Lower bound on any sampling algorithm

Let K be the output of any sampling algorithm between distributions P and Q. Then, with $\alpha = \frac{1}{1+t}$,

$$L(t) \geq D_{\frac{1}{\alpha}}(P||Q) + \frac{\alpha}{1-\alpha}\log(\alpha) - 1. \tag{3}$$

As $t \to 0$ in (3) (resp. $\alpha \to 1$) we recover the lower bound $D(P||Q) - \frac{1}{\ln 2} - 1 \leq \mathbb{E}[l(\mathcal{C}(K))]$.

Result 2: Upper bounds via the Poisson functional representation

Let K be generated by the Poisson functional representation [2]: for $\{U_i\}_{i\geq 1} \sim Q$ and $\{T_i\}_{i\geq 1}$ a rate-one Poisson process, set

$$K = \arg\min_{i\geq 1} \frac{T_i}{\frac{dP}{dQ}(U_i)}. \tag{4}$$

Then, $U_K \sim P$ [2]. We show that, for any $t > 0$ and $\epsilon > 0$, there exists a uniquely decodable encoding of K such that

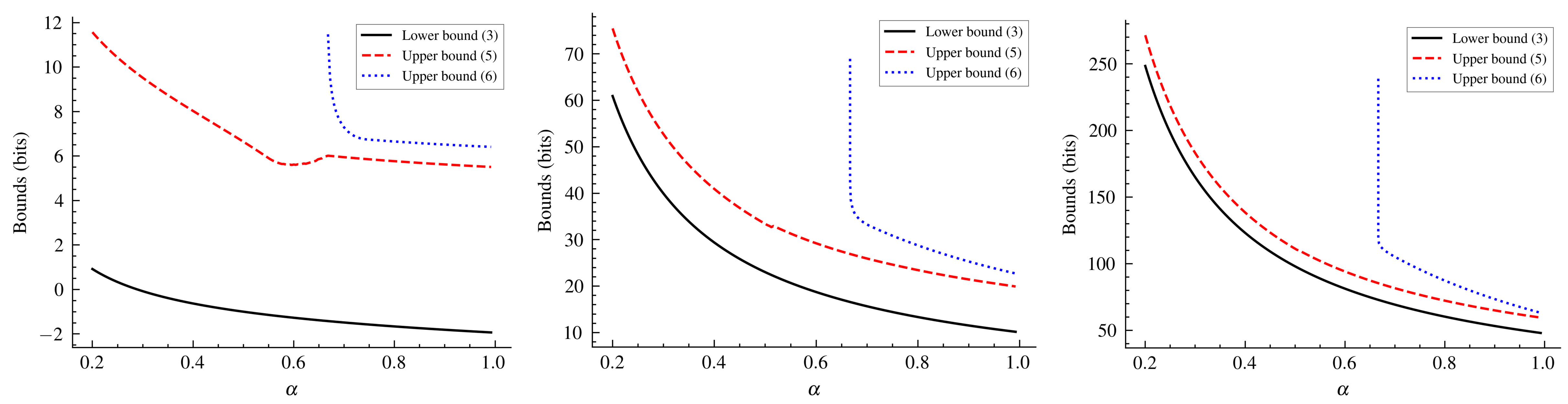
$$L(t) \leq (1+\epsilon) D_{\frac{1+\epsilon(1-\alpha)}{\alpha}}(P||Q) + c(\alpha, \epsilon), \tag{5}$$

with $\alpha = \frac{1}{1+t}$ and $c(\alpha, \epsilon)$ a constant. When $Q = \mathsf{P}_Y$ and $P = \mathsf{P}_{Y|X}(\cdot \mid x)$, (5) upper bounds the exponential cost of channel simulation. If we instead encode K using the Elias omega code [3], for any $2/3 < \alpha < 1$ and $0 < \epsilon \leq \frac{3\alpha-2}{2-2\alpha}$ we have

$$L(t) \leq D_{\frac{2-\alpha}{\alpha}}(P||Q) + (1+\epsilon)\log(D(P||Q)+1) + c(\epsilon), \tag{6}$$

with $c(\epsilon)$ a constant and $\alpha = \frac{1}{1+t}$. As $t \to 0$, (6) reduces to the upper bound of Harsha et al. [4], $\mathbb{E}[l(\mathcal{C}(K))] \leq D(P||Q) + (1+\epsilon)\log((D||P)+1) + c(\epsilon)$. Note that (6) is strictly greater than (5).

Numerical Examples



(a) $P = \mathcal{N}(0,1)$ and $Q = \mathcal{N}(1,1)$. (b) $P = \mathcal{N}(0,1)$ and $Q = \mathcal{N}(5,1)$. (c) $P = \mathcal{N}(0,1)$ and $Q = \mathcal{N}(10,1)$.

Bounds on $L(t)$ for P and Q normal distributions.

The upper and lower bounds are tight within 5-10 bits, even for distributions that are far apart.

References

[1] L. L. Campbell, "A coding theorem and Rényi entropy," Information and control, vol. 8, no. 4, pp. 423-429, 1965.
[2] C. T. Li and A. E. Gamal, "Strong functional representation lemma and applications to coding theorems," in Proc. IEEE International Symposium on Information Theory (ISIT), 2017, pp. 589-593.
[3] P. Elias, "Universal codeword sets and representations of the integers," IEEE Transactions on Information Theory, vol. 21, no. 2, pp.194-203, 2003.
[4] P. Harsha, R. Jain, D. McAllester, and J. Radhakrishnan, "The communication complexity of correlation," IEEE Transactions on Information Theory, vol. 56, no. 1, pp. 438-449, 2010.