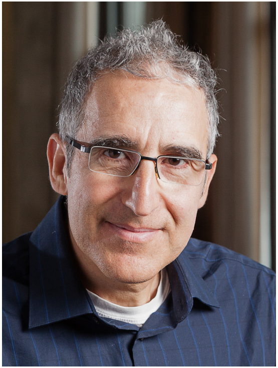
Exact Channel Simulation under Exponential Cost

Spencer Hill

Queen's University, Canada

October 22, 2025

Joint work with



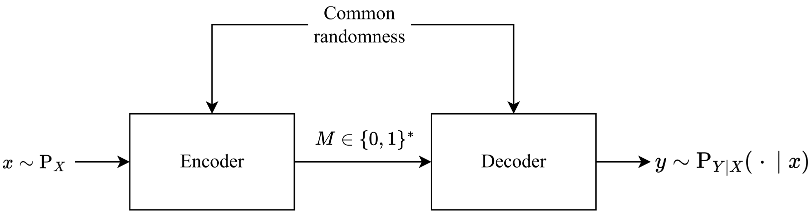
Tamás Linder Fady Alajaji

Queen's University

Outline

1. What is channel simulation?
2. Interesting applications
3. Channel simulation algorithms and performance
4. Exponential (Campbell) cost
5. Our results

Channel Simulation



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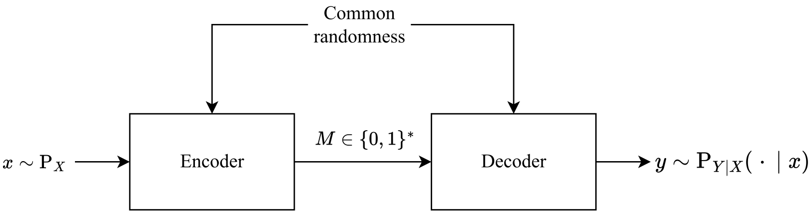

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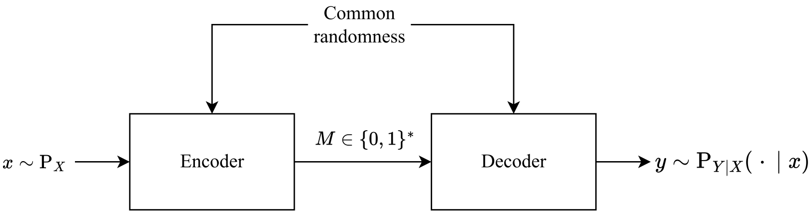
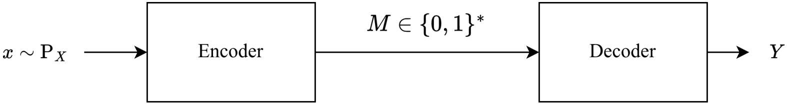


- Use noiseless channel to simulate noisy channel $X \to Y$
- When the goal is to efficiently communicate M, one can achieve

$$\mathbb{E}|M| \approx I(X;Y) \quad \text{bits}$$

Why Care?

Lossy Source Coding



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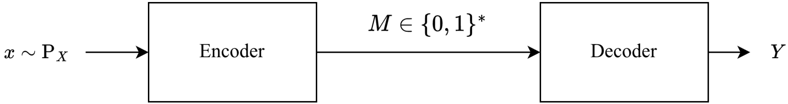

- The encoder encodes the block $(X_1, \ldots, X_n)$

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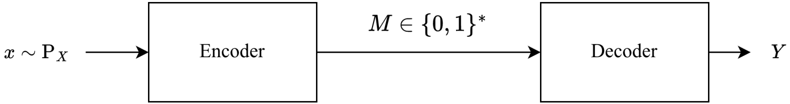

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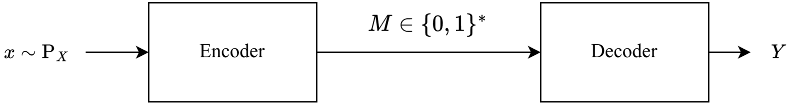

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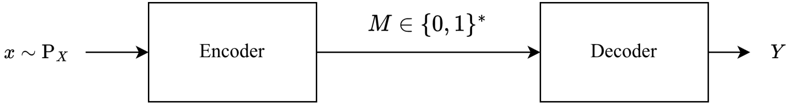

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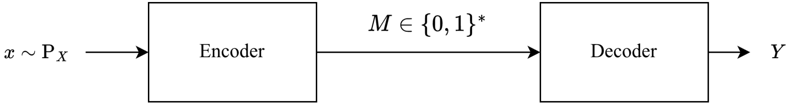

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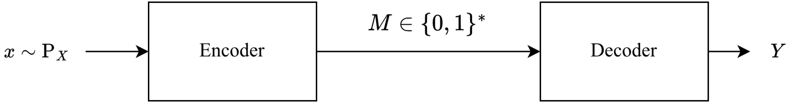
- Rate: $R = \frac{1}{n}\mathbb{E}|M|$ (expected message length)
- Asymptotically $(n \to \infty)$ optimal performance

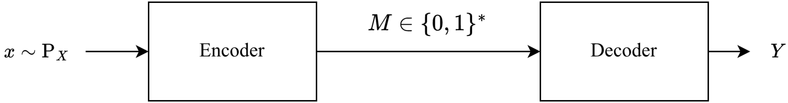
$$R(D) = \min_{\mathrm{P}_{Y|X} \,:\, \mathbb{E}[d(X,Y)] \le D} I(X;Y).$$

The Usual Implementation



1. Quantizer Q

The Usual Implementation



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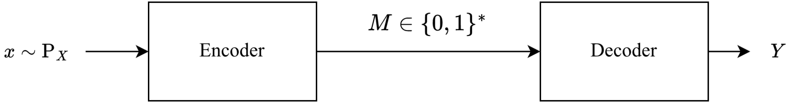

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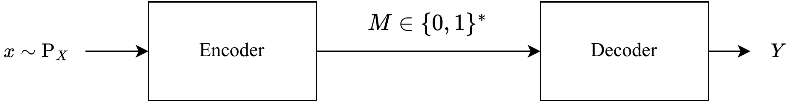

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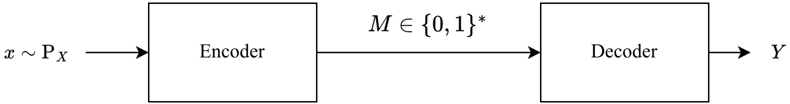

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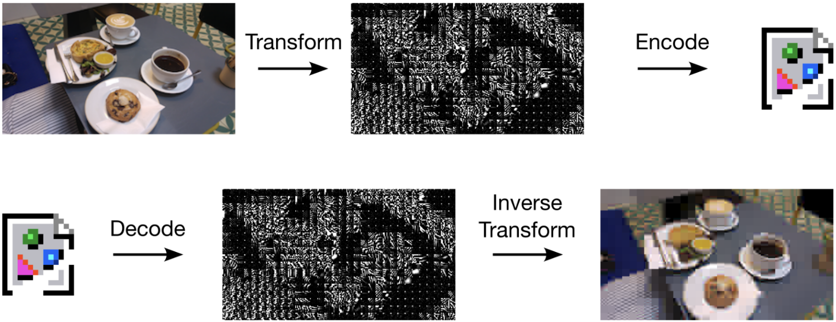
- $M = K_{\hat{P}} \circ Q$
- $Y = K_{\hat{P}}^{-1}(M)$

Transform Coding (JPEG)



Usual transform: $Q \circ f$; for JPEG f is the discrete cosine transform

Neural Transform Coding

Idea: Learn the transform f using a neural network

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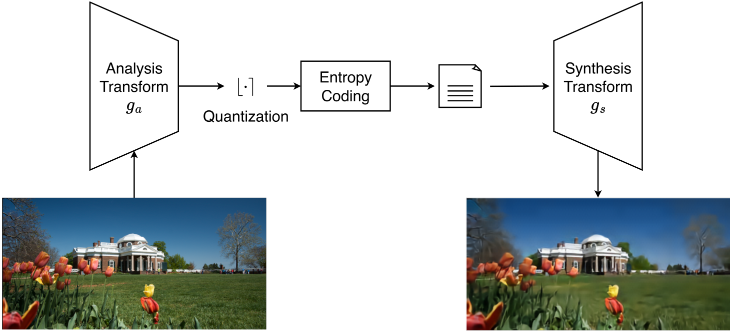

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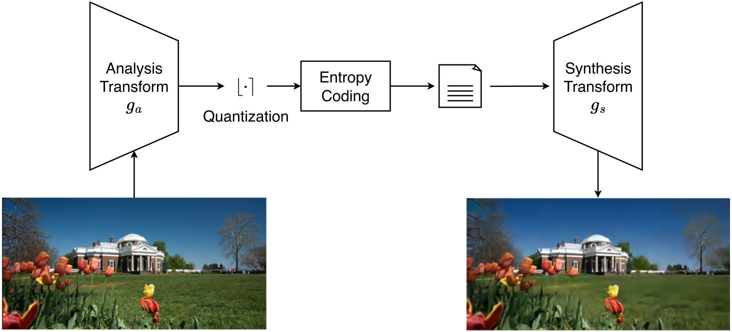

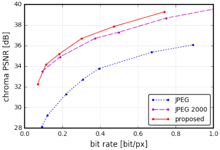

JPEG

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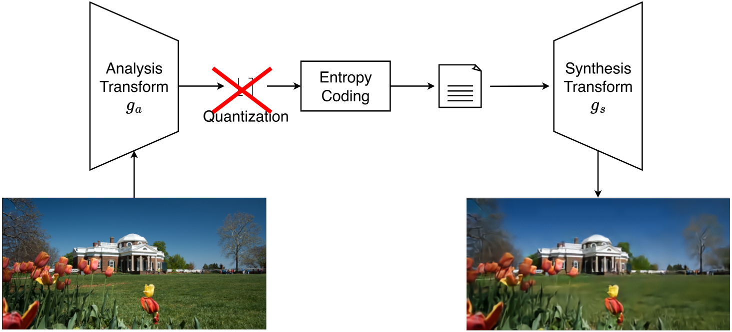
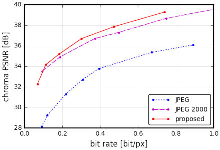
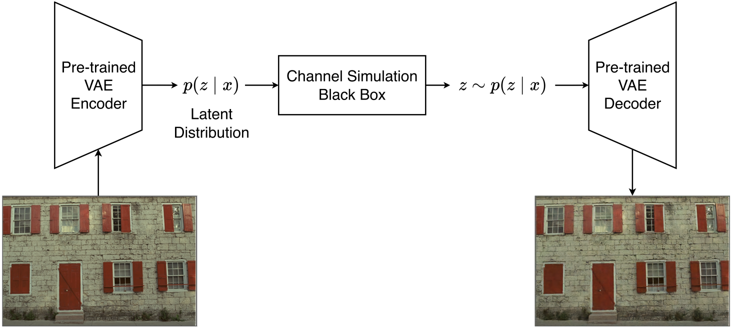




JPEG

Neural Compression with Channel Simulation



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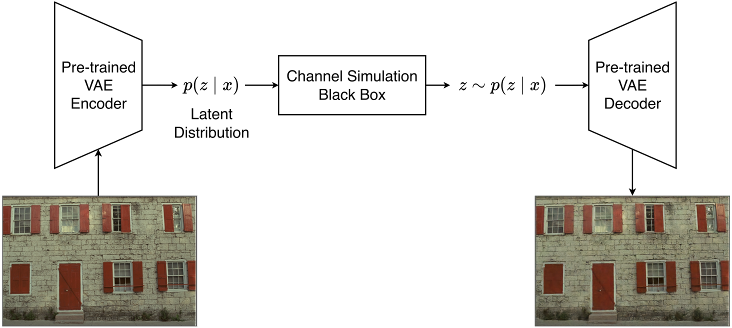

Fully differentiable end-to-end system!

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- Federated learning, …

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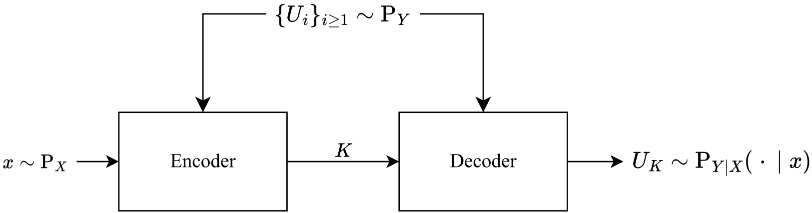

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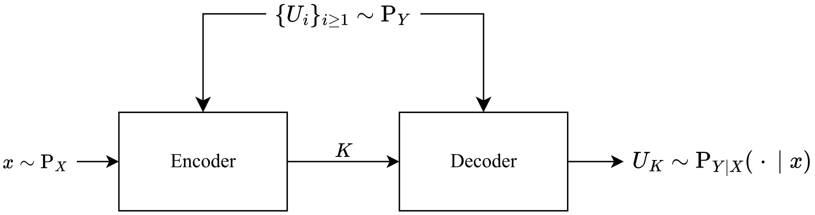

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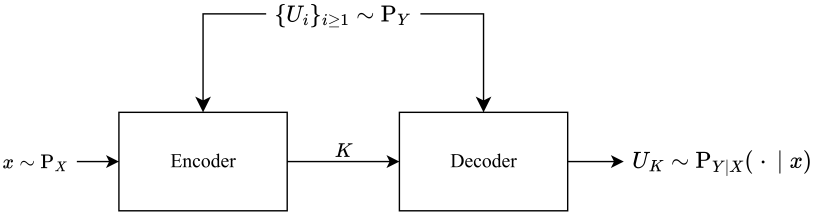

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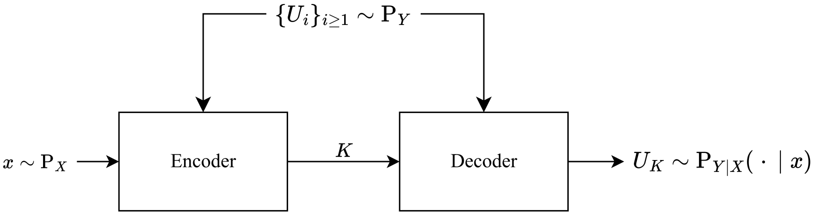

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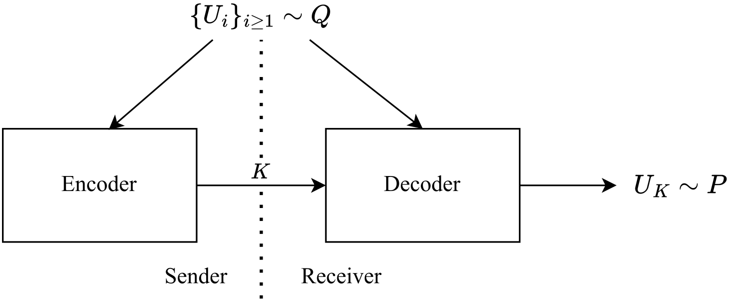
- Sampling can simulate $X \to Y$ with communication cost

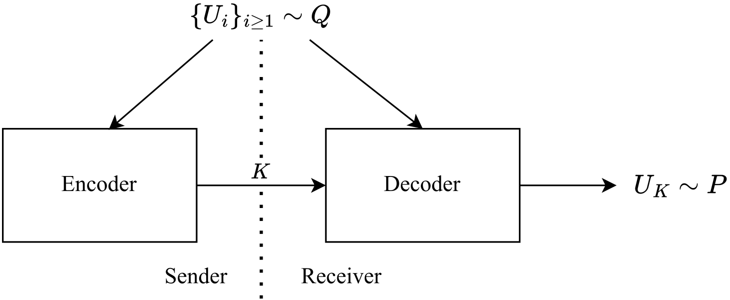
$$\mathbb{E}|M| \approx \mathbb{E}_X[D(\mathrm{P}_{Y|X}(\,\cdot \mid X) \,||\, \mathrm{P}_Y)] = I(X;Y) \quad \text{bits}$$

Exact Sampling



Key Questions:

Exact Sampling



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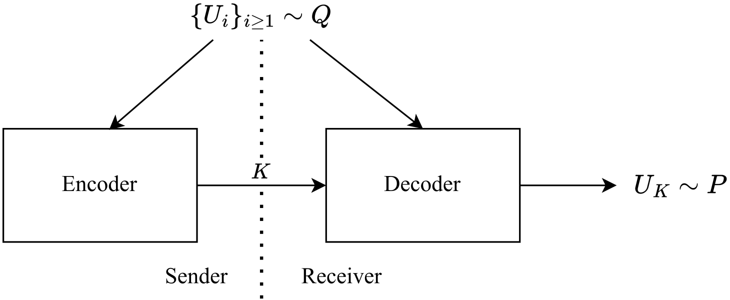
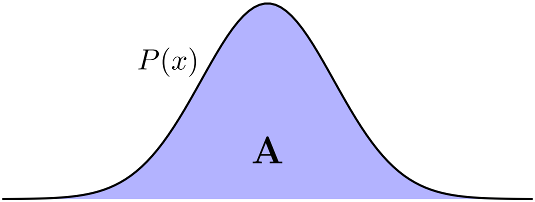


Key Questions:

- How can we choose K such that $U_K \sim P$ *exactly*?
- For *any* sampling algorithm, $D(P||Q) \leq \mathbb{E}[|M|]$. How close can we get to $D(P||Q)$?

Rejection Sampling



- Fact: $(x, y) \sim \text{Unif}(A) \implies x \sim P$.

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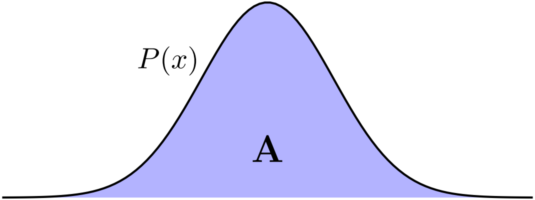

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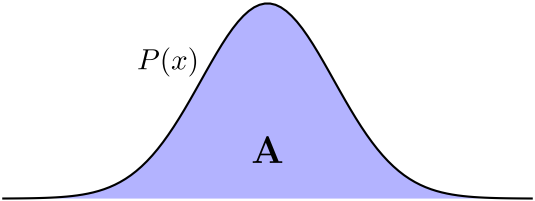

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- For Q not uniform, accept U_k with probability $\gamma \frac{\mathrm{d}P}{\mathrm{d}Q}(U_k)$, $\gamma > 0$ s.t. $\gamma \frac{\mathrm{d}P}{\mathrm{d}Q}(u) \leq 1$ for all u.

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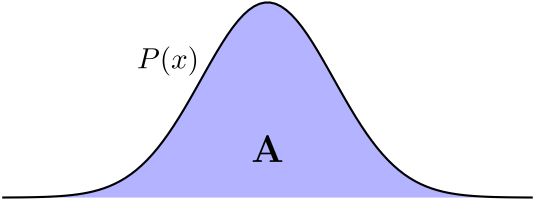

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$$\text{RS: } \mathbb{E}[|M|] \approx D_\infty(P||Q) \gg D(P||Q).$$

Greedy Rejection Sampling

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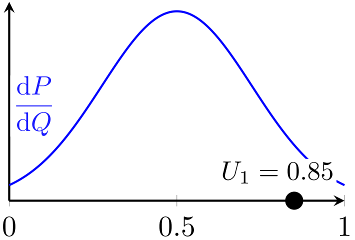
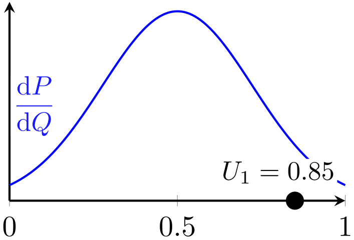


Rejection sampling

$\mathbb{P}(\text{Accept}) = \gamma \frac{dP}{dQ}(U_1) = 0.275$



Greedy rejection sampling

$\mathbb{P}(\text{Accept}) = \left(\frac{dP}{dQ}(U_1) - 0\right)/1 = 0.5$

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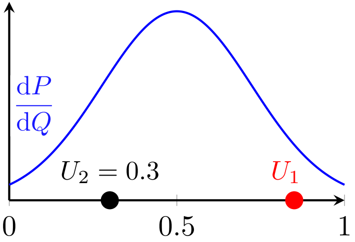

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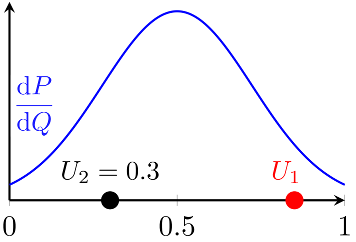

Greedy rejection sampling

$\mathbb{P}(\text{Accept}) = \gamma \frac{\mathrm{d}P}{\mathrm{d}Q}(U_2) = 0.67$ $\mathbb{P}(\text{Accept}) = \frac{1}{0.255}\left(\frac{\mathrm{d}P}{\mathrm{d}Q}(U_2) - 1\right) = 0.89$

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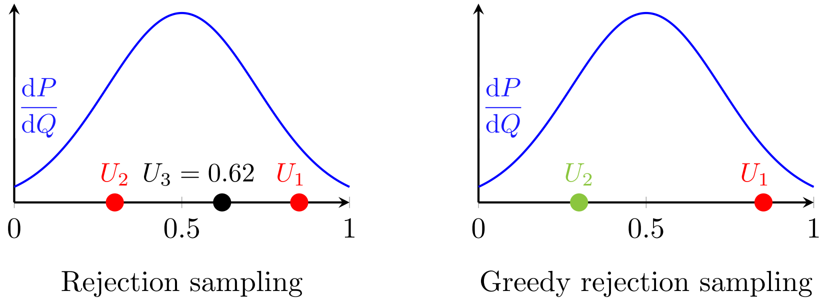

$\mathbb{P}(\text{Accept}) = \gamma \frac{\mathrm{d}P}{\mathrm{d}Q}(U_2) = 0.87$

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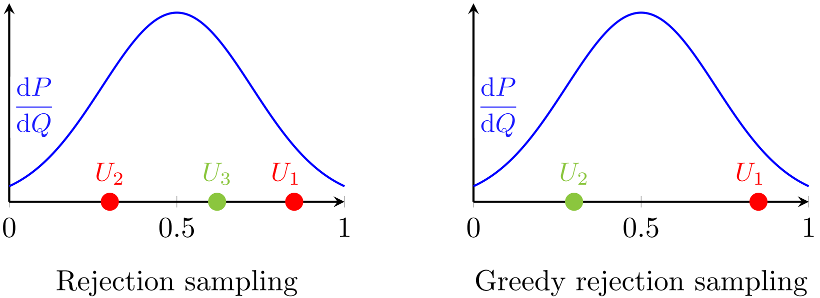

GRS: $\mathbb{E}[|M|] \leq D(P||Q) + \log_2(D(P||Q) + 1) + 4$

Poisson Functional Representation

For $\{T_i\}_{i\geq 1}$ a rate-one Poisson process, choose $K = \arg\min_{i\geq 1} \frac{T_i}{\frac{\mathrm{d}P}{\mathrm{d}Q}(U_i)}$,

Li and El-Gamal (2018).

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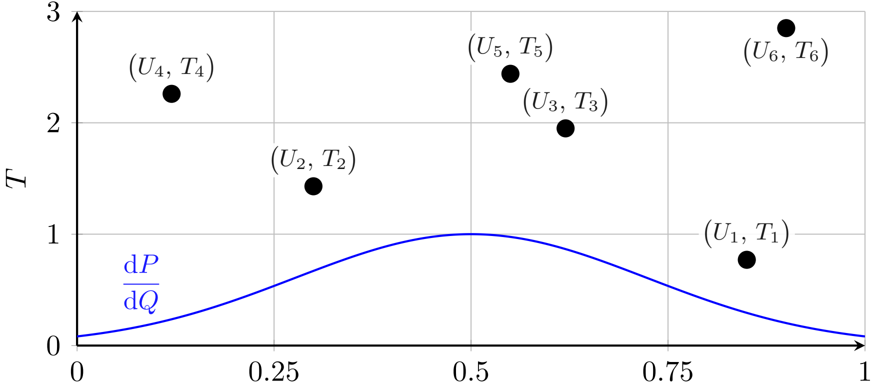

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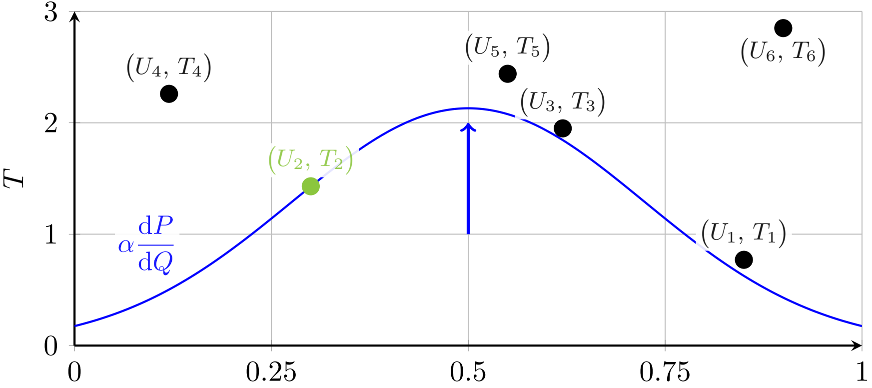

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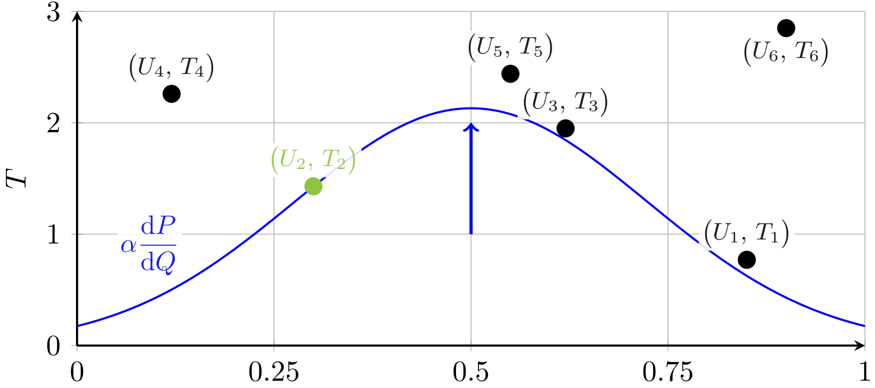

$$\mathbb{E}[|M|] \leq D(P||Q) + \log_2(D(P||Q) + 2) + 3$$

Our Setup: Exponential Cost and Rényi's entropy

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- The previous results are for the expected message length (number of bits) $\mathbb{E}|M|$.
- What are the fundamental limits of exact sampling and channel simulation under a cost which is *exponential* in the message lengths? Can these limits be (almost) achieved by existing algorithms?

Some Definitions

For a discrete random variable K with pmf P_K and any $\alpha \in (0,1) \cup (1,\infty)$, the Rényi entropy $H_\alpha(K)$ is

$$H_\alpha(X) = \frac{1}{1-\alpha} \log\left(\sum_{k \in \mathcal{K}} \mathrm{P}_K(k)^\alpha \right).$$

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Note that $\lim_{\alpha \to 1} H_\alpha(K) = H(K)$ and $\lim_{\alpha \to 1} D_\alpha(P||Q) = D(P||Q)$.

Campbell Cost $L(t)$

For uniquely decodable binary encoding $M \in \{0,1\}^*$ of K having length $|M|$ and for $t > 0$,

$$L(t) = \frac{1}{t} \log\Big(\mathbb{E}[2^{t|M|}]\Big).$$

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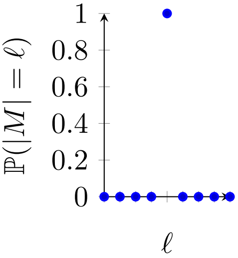
For a random variable K with Rényi entropy $H_\alpha(K)$ encoded optimally into message M, Campbell (1965) showed

$$H_\alpha(K) \le L(t) < H_\alpha(K) + 1$$

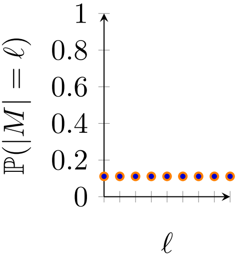
with $\alpha = \frac{1}{1+t}$.

Why Care About $L(t)$?

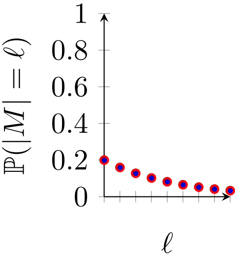
Degenerate at 5



Uniform on $\{1, \ldots, 9\}$



Geometric ($p = 0.2$)



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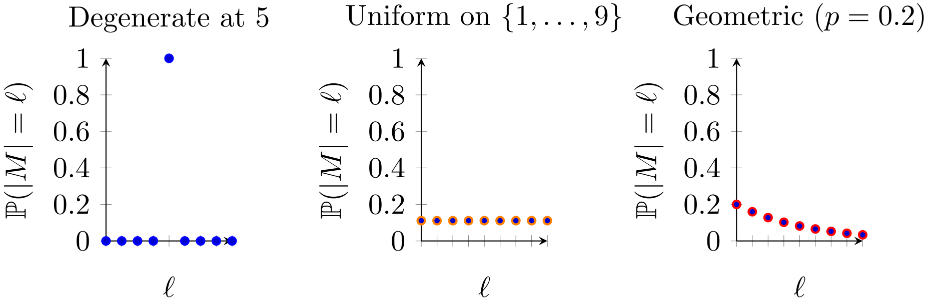

t	**Degenerate** $L(t)$	**Uniform** $L(t)$	**Geometric** $L(t)$
0	5	5	5
0.2	5	5.65	11.83
1	5	7.26	∞
5	5	8.56	∞
∞	5	9	∞

Lower Bound

Theorem 1 For any sampling algorithm and $t > 0$, with $\alpha = \frac{1}{1+t}$,

$$L(t) \geq D_{\frac{1}{\alpha}}(P||Q) + \frac{\alpha}{1-\alpha}\log_2(\alpha) - 1. \tag{1}$$

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As $t \to 0$, we recover the lower bound

$$\mathbb{E}[|M|] \geq D(P||Q) - \frac{1}{\ln(2)} - 1.$$

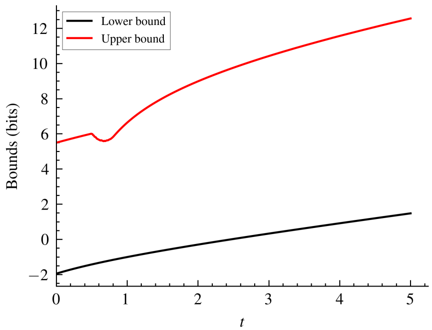
Upper Bound via Poisson Functional Representation

Theorem 2 For K chosen using the Poisson functional representation, for any $\epsilon > 0$ there exists a uniquely decodable encoding of K such that

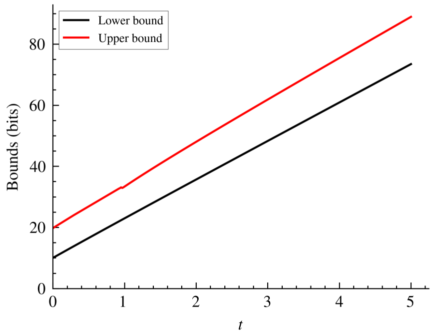
$$L(t) \leq (1+\epsilon) D_{\frac{1+\epsilon(1-\alpha)}{\alpha}}(P||Q) + c(\alpha, \epsilon), \tag{2}$$

with $c(\alpha, \epsilon)$ a constant and $\alpha = \frac{1}{1+t}$.

Gaussian Examples



$P = \mathcal{N}(0,1)$ and $Q = \mathcal{N}(1,1)$



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Proof Techniques

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- For $\mathbb{E}[|M|]$, the lower bound $D(P||Q)$ is relatively simple to prove.
- For $L(t)$ the lower bound $D_{\frac{1}{\alpha}}(P||Q)$ is *not* as simple, requiring a lower bound on $\mathbb{E}[K^t]$ for any sampling algorithm and an argument about injective (not just uniquely decodable) codes.

Asymptotic Results

- We want to use the channel n-times with i.i.d. input $X_1, \dots, X_n$. Thus we sample from the product distribution $P^{\otimes n}$ using samples from $Q^{\otimes n}$.

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Theorem 4 For any $t > 0$, let $L_n^*(t)$ be the *minimum Campbell cost* for target $P^{\otimes n}$ and common randomness $\{U_i\}_{i\geq 1} \sim Q^{\otimes n}$. Then, with $\alpha = \frac{1}{1+t}$,

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This generalizes known results: for the *minimum bits/sample* rate R_n^* for the n-dimensional product distributions,

$$\lim_{n\to\infty} \frac{R_n^*}{n} = D(P||Q).$$

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- A *causal* sampler accepts/rejects each candidate one-at-a-time (K is a stopping time w.r.t. $\{U_i\}_{i\geq 1}$).

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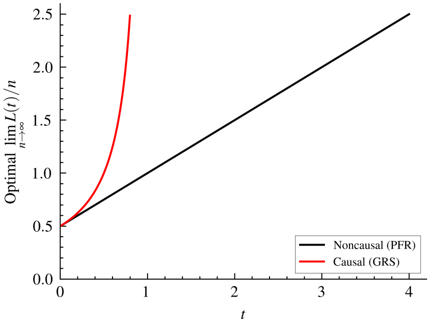
- $D_\beta(P\|Q) > D_{\frac{1}{\alpha}}(P\|Q)$ in general!!!

Asymptotic Gaussian Examples



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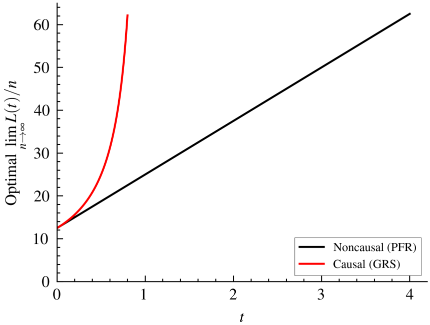

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Greedy rejection sampling does **strictly worse** in the **exponential cost regime**, and the gap is often significant.

Some Open Questions in Channel Simulation

- *Fast* channel simulation algorithms (linear time complexity in n and $D(P||Q)$), especially for P and Q Gaussians.

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- Practical implementation of common randomness

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- The Campbell cost $L(t)$ generalizes the expected message length and can be made more sensitive to the tails of the distribution.
- Under the Campbell cost, the Poisson functional representation is nearly optimal for exact sampling.
- Causal samplers (such as greedy rejection sampling, greedy Poisson rejection sampling, etc.) do **strictly worse** than noncausal samplers in the asymptotic Campbell cost.

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